

# Quantization of Damping Particle Based On New Variational Principles

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In this paper a new approach is proposed to quantize mechanical systems whose equations of motion can not be put into Hamiltonian form. This approach is based on a new type of variational principle, which is adopted to describe a relation: a damping particle may share a common phase curve with a free particle, whose Lagrangian in the new variational principle can be considered as a Lagrangian density in phase space. According to Feynman's theory, the least action principle is adopted to modify the Feynman's path integral formula, where Lagrangian is replaced by Lagrangian density. In the case of conservative systems, the modification reduces to standard Feynman's propagator formula. As an example a particle with friction is analyzed in detail.

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## I. INTRODUCTION

Quantum mechanics is more rigorous than the classical one, which is the best elaborated and understood part of physics. Classical mechanics can be thought of as the base of quantum mechanics. Lagrangian Mechanics and Hamiltonian mechanics are geometrical description of classical mechanics. In physics, quantization is the process of explaining a classical understanding of physical phenomena in terms of a newer understanding known as quantum mechanics. This is a generalization of the procedure for building quantum mechanics from classical mechanics. If a physical phenomenon possesses Lagrangian or Hamiltonian description, the quantization procedure can be easily performed. But for damping classical physical phenomena, it is complicated to perform a quantization procedure from the classical description to the quantum one. Readers can read the complete history of the important ideas in this field in the literature<sup>1</sup>. Recent works on quantization of damped systems see papers<sup>2-8,19</sup>.

The works of Kochan<sup>9,10,11,12</sup> attract our attention, because there is a close link between the works and the classical mechanics. Kochan<sup>9</sup> utilized the tools of contact geometry to give a new picture of classical mechanics, and then defined a new variational principle of least action<sup>12</sup>. Based on the classical picture and the variational principle, a quantum propagator<sup>12</sup> for damping systems was derived.

The ideas of Kochan motivated us to turn our interest to quantum mechanics, because we have found a relation between a damping classical mechanical system and conservative classical ones, which can be described as a variational principle<sup>13</sup>. This picture of classical mechanics is illustrated by a simple example in Sec. II. Therefore, we attempt to utilize our variational principle<sup>13</sup> to construct a new quantum propagator. Our approach to define a new propagator is explained in Sec. III. In Sec. IV a simple example, where the quantization of a particle with friction  $\kappa\dot{x}$  is performed, is reported. In this example, comparison with several other approaches is made.

## II. REVIEW ON A NEW VARIATIONAL PRINCIPLE

### A. A Proposition

To begin, it is necessary to review the new variational principle<sup>13</sup>, which is adopted to describe a relation between damping classical mechanical system and some conservative ones. The relation is represented as a proposition:

**Proposition II.1.** *For any non-conservative classical mechanical system and arbitrary initial condition, there exists a conservative system; both systems sharing one and only one common phase curve; and the value of the Hamiltonian of the conservative system is equal to the sum of the total energy of the non-conservative system on the aforementioned phase curve and a constant depending on the initial condition.*

This proposition can be demonstrated by a simple example. Consider a special one-dimensional simple mechanical system

$$\ddot{x} + \kappa\dot{x} = 0, \quad (1)$$

where  $\kappa$  is a constant. The exact solution of the equation above is

$$x = \Lambda + \eta e^{-\kappa t}, \quad (2)$$

where  $\Lambda, \eta$  are constants. Differentiation gives the velocity:

$$\dot{x} = -\kappa\eta e^{-\kappa t}. \quad (3)$$

From the initial condition  $x_0, \dot{x}_0$ , we find  $\Lambda = x_0 + \dot{x}_0/\kappa$ ,  $\eta = -\dot{x}_0/\kappa$ . Inverting Eq. (2) yields

$$t = -\frac{1}{\kappa} \ln \frac{x - \Lambda}{\eta} \quad (4)$$

and by substituting into Eq. (3), such we have

$$\dot{x} = -\kappa(x - \Lambda) \quad (5)$$

The dissipative force  $F$  in the dissipative system (1) is

$$F = \kappa\dot{x}. \quad (6)$$

Substituting Eq. (5) into Eq. (6), the conservative force  $\mathcal{F}$  is expressed as

$$\mathcal{F} = -\kappa^2(x - \Lambda); \quad (7)$$

Clearly, the conservative force  $\mathcal{F}$  depends on the initial condition of the dissipative system (1), in other words, an initial condition determines a conservative force. Consequently, a new conservative system yields

$$\ddot{x} + \mathcal{F} = 0 \rightarrow \ddot{x} - \kappa^2(x - \Lambda) = 0. \quad (8)$$

The stiffness coefficient in this equation must be negative. One can readily verify that the particular solution (2) of the dissipative system can satisfy the conservative one (8). This point agrees with Proposition (II.1).

The Lagrangian of the system (8) is

$$\hat{L} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\kappa^2 x^2 - \kappa^2 \Lambda x \quad (9)$$

## B. Infinite-dimensional Variational Principle

The ideas of literatures<sup>14–16</sup> have motivated us to regard the mechanical system (1) as a special fluid, which is a collection of fluid particles in phase space. Therefore, first let the label of a particle in the phase space be

$$\mathbf{a} = (x_0, \dot{x}_0); \quad (10)$$

the coordinate of a particle in the configuration space

$$\mathbf{q} = \mathbf{q}(\mathbf{a}, t) = \{x(\mathbf{a}, t), \dot{x}(\mathbf{a}, t)\}; \quad (11)$$

$\rho_o = 1$ . One can consider  $\hat{L}$  in Eq. (9) as a Lagrangian density of the system (1)

$$\mathcal{L} = \hat{L} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\kappa^2 x^2 - \kappa^2 \Lambda x \quad (12)$$

Thus the Lagrangian functional of Eq. (1) can be presented as the following:

$$L[x, \dot{x}] = \int_D \mathcal{L} d^2 \mathbf{a} = \int_D \left[ \frac{1}{2}\dot{x}^2 + \frac{1}{2}\kappa^2 x^2 - \kappa^2 \Lambda x \right] d^2 \mathbf{a}, \quad (13)$$

where  $d^2 = dx_0 d\dot{x}_0$ . Thus the action functional can be presented as follows:

$$S[\mathbf{q}] = \int_{t_0}^{t_1} L[x, \dot{x}] dt = \int_{t_0}^{t_1} dt \int_D \left[ \frac{1}{2}\dot{x}^2 + \frac{1}{2}\kappa^2 x^2 - \kappa^2 \Lambda x \right] d^2 \mathbf{a} \quad (14)$$

According to Hamiltonian theorem, we have the functional derivative  $\delta S/\delta \mathbf{q}(a, t) = 0$ ,

$$\begin{aligned} \frac{\delta S}{\delta \mathbf{q}(\mathbf{a}, t)} = 0 &\implies \frac{\delta L}{\delta \mathbf{q}(\mathbf{a}, t)} - \frac{d}{dt} \frac{\delta L}{\delta \dot{\mathbf{q}}(\mathbf{a}, t)} = 0 \\ &\implies \frac{\partial \mathcal{L}}{\partial \mathbf{q}(\mathbf{a}, t)} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}(\mathbf{a}, t)} = -\ddot{x}(\mathbf{a}, t) - \kappa^2(x - \Lambda) = 0. \end{aligned} \quad (15)$$

The equation above implies that subject to the initial condition  $\mathbf{a}$ , an associated conservative system exists, the control equation of which is Eq. (8), the phase curve of which coincides with that of the damping system (1). In the case of the classical least action principle, the stationary curves set is represented as an Euler-Lagrangian differential equation, which can be converted into a uniform conservative Newtonian equation. In the case of our least action principle, the stationary curves set is represented as an Euler-Lagrangian functional equation, which can be converted varied conservative Newtonian equation subject to varied initial condition  $\mathbf{a}$ .

Furthermore we can define an action density  $\hat{S}$ , such that Eq. (14) can be rewritten as

$$S[\mathbf{q}] = \int_D \hat{S} d^2 \mathbf{a} = \int_D d^2 \mathbf{a} \int_{t_0}^{t_1} \mathcal{L} dt, \quad (16)$$

where  $\hat{S} = \int_{t_0}^{t_1} \mathcal{L} dt$ . If Eq. (1) is conservative, the action density reduces back to the classical action, and the Lagrangian density back to a uniform classical Lagrangian, our least action principle back to the classical least action principle.

### III. DERIVATION OF A QUANTUM PROPAGATOR

According to Feynman's theory<sup>17</sup>, the probability amplitude of the transition of a system from the space-time configuration  $(x_a, 0)$  to another space-time configuration  $(x_b, T)$  is

$$\begin{aligned} K(x_b, T; x_a, 0) &= \frac{1}{N} \int [\mathcal{D}\gamma] \exp \left\{ \frac{i}{\hbar} S \right\} \\ &= \frac{1}{N} \int [\mathcal{D}\gamma] \exp \left\{ \frac{i}{\hbar} \int_{\gamma} L dt \right\} \end{aligned} \quad (17)$$

Here the integral is taken over all pathes  $\gamma(t) = (x, t)$  or all pathes in the phase space, satisfying the boundary conditions

$$x(0) = x_a, \quad x(T) = x_b. \quad (18)$$

Kochan employed his variational principle to replace  $Ldt$  in Feynman's kernel function with a Lepage two-form  $\Omega$ . Motivated by the Kochan's trick<sup>12</sup>, encouraged by the sentence from

Feynman's thesis<sup>18</sup> too: “the central mathematical concept is the analogue of the action in classical mechanics. It is therefore applicable to mechanical systems whose equations of motion cannot be put into Hamiltonian form. It is only required that some sort of least action principle be available”, we propose a method to generalize the Feynman's probability amplitude:

$$\begin{aligned} K(x_b, T; x_a, 0) &= \frac{1}{N} \int [\mathcal{D}\gamma] \exp \left\{ \frac{i}{\hbar} \hat{S} \right\} \\ &= \frac{1}{N} \int [\mathcal{D}\gamma] \exp \left\{ \frac{i}{\hbar} \int_{\gamma} \mathcal{L} dt \right\}, \end{aligned} \quad (19)$$

where the action  $S$  is replaced by the action density  $\hat{S}$  and the Lagrangian  $L$  replaced by the Lagrangian density  $\mathcal{L}$ , the classical least action principle is replaced by the variational principle in Sec. III.

In conservative case, the Lagrangian density  $\mathcal{L}$  reduces back to standard Lagrangian, the propagator (19) reduces back to standard Feynman's propagator (17)

#### IV. EXAMPLE OF QUANTIZATION

Consider the quantization of the system (1) with the boundary condition (18). From the boundary condition can be derived

$$\Lambda = x_a - \frac{x_b - x_a}{e^{-\kappa T} - 1}, \quad \eta = \frac{x_b - x_a}{e^{-\kappa T} - 1} \quad (20)$$

We take Feynman's general method<sup>17</sup> to integrate Eq. (19). First we construct a short-time propagator:

$$\begin{aligned} K(x_{j+1}, x_j; \epsilon) &= \sqrt{\frac{1}{2\pi i \hbar}} \exp \left\{ \frac{i\epsilon}{\hbar} \left[ \frac{(x_{j+1} - x_j)^2}{2\epsilon^2} + \frac{\kappa^2}{4} (x_{j+1}^2 + x_j^2) - \kappa^2 \Lambda \left( \frac{x_{j+1} + x_j}{2} \right) \right] \right\} \\ &= \sqrt{\frac{1}{2\pi i \hbar}} \exp \left\{ \frac{i}{\hbar} [a_0(x_{j+1}^2 + x_j^2) - 2b_0 x_{j+1} x_j - R_0 x_{j+1} - S_0 x_j] \right\}, \end{aligned} \quad (21)$$

where

$$a_0 = \frac{1}{2\epsilon} \left( \frac{1}{\epsilon} + \frac{(\kappa\epsilon)^2}{2} \right), \quad b_0 = \frac{1}{2\epsilon}, \quad R_0 = \frac{\kappa^2 \Lambda \epsilon}{2}, \quad S_0 = \frac{\kappa^2 \Lambda \epsilon}{2}. \quad (22)$$

Substituting the short-time propagator into

$$K(x_b, T; x_a, 0) = \prod_{k=1}^{N-1} \int_{-\infty}^{+\infty} dx_k \prod_{j=0}^{N-1} K(x_{j+1}, x_j; \epsilon),$$

we have

$$\begin{aligned}
K(x_b, T; x_a, 0) &= \lim_{\substack{N \rightarrow \infty \\ \epsilon \rightarrow 0}} \left( \frac{1}{2\pi i \hbar \epsilon} \right)^{N/2} \left[ \prod_{j=1}^{N-1} \int dx_j \right] \\
&\quad \times \exp \left\{ \frac{i}{\hbar} \sum_{j=0}^{N-1} [a_0(x_{j+1}^2 + x_j^2) - 2b_0 x_{j+1} x_j - R_0 x_{j+1} - S_0 x_j] \right\} \\
&= \lim_{N \rightarrow \infty} A_1 \left[ \prod_{j=1}^{N-1} \int dx_j \right] \exp \left\{ \frac{i}{\hbar} \phi_1 \right\} \\
&= \lim_{N \rightarrow \infty} A_2 \left[ \prod_{j=2}^{N-1} \int dx_j \right] \exp \left\{ \frac{i}{\hbar} \phi_2 \right\} \\
&= \lim_{N \rightarrow \infty} A_k \left[ \prod_{j=k}^{N-1} \int dx_j \right] \exp \left\{ \frac{i}{\hbar} \phi_k \right\} \\
&= \lim_{N \rightarrow \infty} A_N \exp \left\{ \frac{i}{\hbar} \phi_N \right\}. \tag{23}
\end{aligned}$$

Here

$$\begin{aligned}
A_1 &= \left( \frac{1}{2\pi i \hbar \epsilon} \right)^{N/2} \\
\phi_1 &= \sum_{j=2}^{N-1} [a_0(x_{j+1}^2 + x_j^2) - 2b_0 x_{j+1} x_j] + \alpha_1 \\
\alpha_1 &= a_0(x_2^2 + x_1^2) - 2b_0 x_2 x_1 - R_0 x_2 - S_0 x_1 + a_0(x_1^2 + x_0^2) - 2b_0 x_1 x_0 - R_0 x_1 - S_0 x_0 \\
&= a_0(x_2^2 + x_0^2) - R_0 x_2 - S_0 x_0 + 2a_0 \left[ x_1 - \frac{b_0(x_2 + x_0)^2 + (S_0 + R_0)/2}{2a_0} \right]^2 - \\
&\quad \frac{[b_0(x_2 + x_0)^2 + (S_0 + R_0)/2]^2}{2a_0} \\
&= 2a_0 \left[ x_1 - \frac{b_0(x_2 + x_0)^2 + (S_0 + R_0)/2}{2a_0} \right]^2 + a_1(x_2^2 + x_0^2) - 2b_1 x_2 x_0 - R_1 x_2 - S_1 x_0 - \Omega_1
\end{aligned}$$

where

$$a_1 = a_0 - \frac{b_0^2}{2a_0}, \quad b_1 = \frac{b_0}{2a_0}, \tag{24}$$

$$R_1 = R_0 + \frac{b_0(S_0 + R_0)}{2a_0} \quad S_0 = S_0 + \frac{b_0(S_0 + R_0)}{2a_0} \quad \Omega_1 = \frac{(R_0 + S_0)^2}{8a_0} \tag{25}$$

Integration is first done with respect to  $x_1$ , we have

$$A_2 = A_1 \left( \frac{i\pi \hbar}{2a_0} \right)^{1/2} \exp \left( -\frac{i}{\hbar} \Omega_1 \right)$$

$$\begin{aligned}
\phi_2 &= \sum_{j=2}^{N-1} [a_0(x_{j+1}^2 + x_j^2) - 2b_0x_{j+1}x_j] + a_1(x_2^2 + x_0^2) - 2b_1x_2x_0 - R_1x_2 - S_1x_0 \\
\phi_k &= \sum_{j=k}^{N-1} [a_0(x_{j+1}^2 + x_j^2) - 2b_0x_{j+1}x_j] + a_{k-1}(x_k^2 + x_0^2) - 2b_{k-1}x_kx_0 - R_{k-1}x_k - S_{k-1}x_0 \\
&= \sum_{j=k+1}^{N-1} [a_0(x_{j+1}^2 + x_j^2) - 2b_0x_{j+1}x_j] + \alpha_k,
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
\alpha_k &= a_0(x_{k+1}^2 + x_k^2) - 2b_0x_{k+1}x_k - R_0X_{k+1} - S_0x_k \\
&\quad + a_{k-1}(x_k^2 + x_0^2) - 2b_{k-1}x_kx_0 - R_{k-1}X_k - S_{k-1}x_0 \\
&= a_0x_{k+1}^2 + a_{k-1}x_0^2 - R_0X_{k+1} - S_{k-1}x_0 + \frac{1}{a_0 + a_{k-1}} \left[ x_k - \frac{(b_0x_{k+1} + b_{k-1}x_0) + (R_{k-1} + S_0)/2}{a_0 + a_{k-1}} \right]^2 \\
&\quad - \frac{[(b_0x_{k+1} + b_{k-1}x_0) + (R_{k-1} + S_0)/2]^2}{a_0 + a_{k-1}}
\end{aligned}$$

Integration is first done with respect to  $x_k$ , we have

$$A_{k+1} = A_k \sqrt{\frac{i\pi\hbar}{a_0 + a_{k-1}}} \exp\left(-\frac{i}{\hbar}\Omega_k\right)$$

$$\begin{aligned}
\phi_{k+1} &= \sum_{j=k+1}^{N-1} [a_0(x_{j+1}^2 + x_j^2) - 2b_0x_{j+1}x_j] + a_0x_{k+1}^2 + a_{k-1}x_0^2 - R_0X_{k+1} - S_{k-1}x_0 \\
&\quad - \frac{[(b_0x_{k+1} + b_{k-1}x_0) + (R_{k-1} + S_0)/2]^2}{a_0 + a_{k-1}} \\
&= \sum_{j=k+1}^{N-1} [a_0(x_{j+1}^2 + x_j^2) - 2b_0x_{j+1}x_j] \\
&\quad + a_kx_{k+1}^2 + a'_kx_0^2 - 2b_kx_{k+1}x_0 - R_kx_{k+1} - S_kx_0 - \Omega_k,
\end{aligned} \tag{27}$$

where

$$a_k = a_0 - \frac{b_0^2}{a_0 + a_{k-1}}, \quad a'_k = a_{k-1} - \frac{b_{k-1}^2}{a_0 + a_{k-1}}, \quad b_k = \frac{b_0b_{k-1}}{a_0 + a_{k-1}}, \tag{28}$$

$$R_k = R_0 + \frac{(R_{k-1} + S_0)b_0}{a_0 + a_{k-1}}, \quad S_k = S_{k-1} + \frac{(R_{k-1} + S_0)b_{k-1}}{a_0 + a_{k-1}}, \quad \Omega_k = \frac{(R_{k-1} + S_0)^2}{4(a_0 + a_{k-1})} \tag{29}$$

If  $a_k = a'_k$ , i.e.

$$a_{k-1}^2 = b_{k-1}^2 + a_0^2 - b_0^2, \tag{30}$$



then the form of  $\phi_{k+1}$  is same as the form of  $\phi_k$ . Eq. (30) is tenable as  $k = 1, k = 2$ , because from Eq. (25) can be derived

$$a_1^2 - b_1^2 = \left(a_0 - \frac{b_0^2}{2a_0}\right)^2 - \left(\frac{b_0^2}{2a_0}\right)^2 = a_0^2 - b_0^2.$$

Eq. (27) and Eq. (28) can be proved by mathematical induction for  $k = 1, 2, 3, \dots$

$$A_N = A_{N-1} \sqrt{\frac{i\pi\hbar}{a_0 + a_{N-2}}} \exp\left(-\frac{i}{\hbar}\Omega_{N-1}\right) = \left(\frac{1}{2\pi i\hbar\epsilon}\right)^{N/2} \exp\left(\sum_{k=1}^{N-1} -\frac{i}{\hbar}\Omega_k\right) \prod_{k=1}^{N-1} \sqrt{\frac{i\pi\hbar}{a_0 + a_{k-1}}}$$

$$\phi_N = a_{N-1}(x_N^2 + x_0^2) - 2b_{N-1}x_Nx_0 - R_{N-1}x_N - S_{N-1}x_0$$

Substituting the equation above into Eq. (23), we have

$$K(x_b, T; x_a, 0) = \lim_{\substack{N \rightarrow \infty \\ \epsilon \rightarrow 0}} \left(\frac{1}{2\pi i\hbar\epsilon}\right)^{1/2} \left[\prod_{k=1}^{N-1} \left(\frac{1}{2\epsilon} \frac{1}{a_0 + a_{k-1}}\right)\right]^{1/2} \exp\left(\sum_{k=1}^{N-1} -\frac{i}{\hbar}\Omega_{k-1}\right) \\ \times \exp\left\{\frac{i}{\hbar} [a_{N-1}(x_N^2 + x_0^2) - 2b_{N-1}x_Nx_0 - R_{N-1}x_N - S_{N-1}x_0]\right\} \quad (31)$$

The coefficients  $a_k, b_k$  of Eq. (31) can be derived from the following recursion formulas:

$$a_0 = b_0 \left(1 + 2\left(\frac{\kappa\epsilon}{2}\right)^2\right), \quad b_0 = \frac{1}{2\epsilon} \quad (32)$$

$$a_{k-1} = (b_{k-1}^2 + a_0^2 - b_0^2)^{1/2}, \quad b_k = \frac{b_0 b_{k-1}}{a_0 + a_{k-1}} \quad (33)$$

In order to obtain the result of Eq. (31) with  $N \rightarrow \infty$ , one must obtain the limit of  $a_k, b_k, R_k, S_k, \Omega_k$ . Let us consider  $a_k, b_k$  first. As  $N \rightarrow \infty$ , we have

$$\frac{1}{2}\kappa\epsilon = \sinh \frac{1}{2}\kappa\epsilon. \quad (34)$$

Substituting Eq. (34) into Eq. (32), we have

$$a_0 = b_0 \left(1 + 2 \sinh^2 \frac{1}{2}\kappa\epsilon\right) = b_0 \cosh \kappa\epsilon$$

Substituting the equation above into Eq. (33), we have

$$\frac{1}{b_k} = \frac{\cosh \kappa\epsilon}{b_{k-1}} + \frac{1}{b_0} \sqrt{1 + \frac{b_0^2}{b_{k-1}^2} \sinh \kappa\epsilon} \quad (35)$$

Let  $k = 1$ , Eq. (35) becomes

$$\frac{1}{b_1} = \frac{1}{b_0} (\cosh \kappa\epsilon + \sqrt{1 - \sinh^2 \kappa\epsilon}) = \frac{2 \cosh \kappa\epsilon}{b_0} = \frac{\sinh 2\kappa\epsilon}{b_0 \sinh \kappa\epsilon}. \quad (36)$$

Let  $k = 2$ , we have

$$\frac{1}{b_2} = \frac{\cosh \kappa \epsilon}{b_1} + \frac{1}{b_0} \sqrt{1 + \frac{b_0^2}{b_1^2} \sinh^2 \kappa \epsilon} = \frac{\sinh 3\kappa \epsilon}{b_1 \sinh 2\kappa \epsilon} = \frac{\sinh 3\kappa \epsilon}{b_0 \sinh \kappa \epsilon}$$

The rest may be deduced by analogy, we have

$$\frac{1}{b_{k-1}} = \frac{\sinh k\kappa \epsilon}{b_0 \sinh \kappa \epsilon}$$

The equation above can be proved by recursion method. Substituting the equation above into Eq. (35), we have

$$\begin{aligned} \frac{1}{b_{k-1}} &= \frac{\cosh \kappa \epsilon \sinh k\kappa \epsilon}{b_0 \sinh \kappa \epsilon} + \frac{1}{b_0} \sqrt{1 + b_0^2 \left( \frac{\sinh k\kappa \epsilon}{b_0 \sinh \kappa \epsilon} \right)^2 \sinh^2 \kappa \epsilon} \\ &= \frac{\sinh(k+1)\kappa \epsilon}{b_0 \sinh \kappa \epsilon} \end{aligned}$$

Hence

$$b_k = \frac{1}{2\epsilon} \frac{\sinh \kappa \epsilon}{\sinh(k+1)\kappa \epsilon} \quad (37)$$

$$a_k = (b_k^2 + a_0^2 - b_0^2)^{1/2} = b_k \sqrt{1 + \frac{b_0^2}{b_k^2} \sinh^2 \kappa \epsilon} = \frac{\sinh \kappa \epsilon \cosh(k+1)\kappa \epsilon}{2\epsilon \sinh(k+1)\kappa \epsilon} \quad (38)$$

$$\begin{aligned} a_0 + a_{k-1} &= \frac{1}{2\epsilon} \left( \cosh \kappa \epsilon + \sinh \kappa \epsilon \frac{\cosh k\kappa \epsilon}{\sinh k\kappa \epsilon} \right) \\ &= \frac{1}{2\epsilon} \frac{\sinh(k+1)\kappa \epsilon}{\sinh \kappa \epsilon} \end{aligned} \quad (39)$$

Substituting the equation above into Eq. (29) and let  $k = 1$ , we have

$$\begin{aligned} R_1 &= R_0 + \frac{(R_0 + S_0)b_0}{a_0 + a_0} = \frac{\kappa^2 \Lambda \epsilon}{2} \left( 1 + \frac{1}{\cosh \kappa \epsilon} \right) \\ S_1 &= S_0 + \frac{(R_0 + S_0)b_0}{a_0 + a_0} = \frac{\kappa^2 \Lambda \epsilon}{2} \left( 1 + \frac{1}{\cosh \kappa \epsilon} \right) \end{aligned}$$

Let  $k = 2$  and substitute Eq. (39) into Eq. (29), we have

$$\begin{aligned} R_2 &= R_0 + \frac{(R_1 + S_0)b_0}{a_0 + a_1} = \frac{\kappa^2 \Lambda \epsilon}{2} \left[ 1 + \frac{1}{\cosh \kappa \epsilon} \frac{(2 \cosh \kappa \epsilon + 1) \sinh 2\kappa \epsilon}{\sinh 3\kappa \epsilon} \right] \\ S_2 &= S_1 + \frac{R_1 + S_0}{a_0 + a_1} b_1 = \frac{\kappa^2 \Lambda \epsilon}{2} \left\{ 1 + \frac{1}{\cosh \kappa \epsilon} \left[ 1 + \frac{(2 \cosh \kappa \epsilon + 1) \sinh \kappa \epsilon}{\sinh 3\kappa \epsilon} \right] \right\} \end{aligned}$$

For  $k = 3$ , we have

$$\begin{aligned}
R_3 &= R_0 + \frac{R_2 + S_0}{a_0 + a_2} b_0 = \frac{\kappa^2 \Lambda \epsilon}{2} \left[ 1 + \frac{1}{\cosh \kappa \epsilon} \frac{2 \cosh \kappa \epsilon \sinh 3 \kappa \epsilon + (2 \cosh \kappa \epsilon + 1) \sinh 2 \kappa \epsilon}{\sinh 4 \kappa \epsilon} \right] \\
S_3 &= S_2 + \frac{R_2 + S_0}{a_0 + a_2} b_2 \\
&= \frac{\kappa^2 \Lambda \epsilon}{2} \left\{ 1 + \frac{1}{\cosh \kappa \epsilon} \left[ 1 + (2 \cosh(\kappa \epsilon + 1) \frac{\sinh \kappa \epsilon}{\sinh 3 \kappa \epsilon} \right. \right. \\
&\quad \left. \left. + \frac{2 \cosh \kappa \epsilon \sinh 3 \kappa \epsilon + (2 \cosh \kappa \epsilon + 1) \sinh 2 \kappa \epsilon}{\sinh 3 \kappa \epsilon} \frac{\sinh \kappa \epsilon}{\sinh 4 \kappa \epsilon} \right] \right\}
\end{aligned}$$

We can suppose

$$R_k = \frac{\kappa^2 \Lambda \epsilon}{2} \left[ 1 + \frac{1}{\cosh \kappa \epsilon} \frac{\sum_{j=1}^k 2 \cosh \kappa \epsilon \sinh j \kappa \epsilon}{\sinh(k+1) \kappa \epsilon} \right] \quad (40)$$

$$S_k = \frac{\kappa^2 \Lambda \epsilon}{2} \left\{ 1 + \frac{1}{\cosh \kappa \epsilon} \sum_{j=1}^k \frac{\sinh \kappa \epsilon}{\sinh(j+1) \kappa \epsilon} \left[ \frac{\sum_{l=1}^j 2 \cosh \kappa \epsilon \sinh l \kappa \epsilon}{\sinh j \kappa \epsilon} \right] \right\} \quad (41)$$

Substituting Eq. (40) into the last term in Eq. (29), we have

$$\Omega_k = \frac{\kappa^3 \Lambda^2 \epsilon^3}{4} \frac{\sinh \kappa \epsilon}{\sinh(k+1) \kappa \epsilon} \left[ 2 + \frac{1}{\cosh \kappa \epsilon} \frac{\sum_{j=1}^{k-1} 2 \cosh \kappa \epsilon \sinh j \kappa \epsilon}{\sinh k \kappa \epsilon} \right]^2 \quad (42)$$

Therefore, we have the limit of  $R_k, S_k, \Omega_k$

$$\lim_{N \rightarrow \infty} R_N = \frac{\kappa^2 \Lambda}{2} \frac{e^{2\kappa T} - 2e^{\kappa T} + 1}{\kappa e^{\kappa T} \sinh \kappa T} = \kappa \Lambda \frac{e^{\kappa T} - 1}{e^{\kappa T} + 1} \quad (43)$$

$$\lim_{N \rightarrow \infty} S_N = \kappa \Lambda \frac{e^{\kappa T} - 1}{e^{\kappa T} + 1} \quad (44)$$

$$\lim_{N \rightarrow \infty} \sum_{k=1}^{N-1} \Omega_{k-1} = 0 \quad (45)$$

Substituting Eq. (38) and Eq. (37), Eq. (39), Eq. (43), Eq. (43), Eq. (45) into Eq. (23),

we have

$$\begin{aligned}
K(x_b, T; x_a, 0) &= \lim_{\substack{N \rightarrow \infty \\ \epsilon \rightarrow 0}} \left( \frac{1}{2\pi i \hbar \epsilon} \right)^{1/2} \left[ \prod_{k=1}^{N-1} \left( \frac{1}{2\epsilon} \frac{1}{a_0 + a_{k-1}} \right) \right]^{1/2} \exp \left( \sum_{k=1}^{N-1} -\frac{i}{\hbar} \Omega_{k-1} \right) \\
&\quad \times \exp \left\{ \frac{i}{\hbar} [a_{N-1}(x_N^2 + x_0^2) - 2b_{N-1}x_N x_0 - R_{N-1}x_N - S_{N-1}x_0] \right\} \\
&= \lim_{\substack{N \rightarrow \infty \\ \epsilon \rightarrow 0}} \left( \frac{1}{2\pi i \hbar \epsilon} \right)^{1/2} \left[ \prod_{k=1}^{N-1} \left( \frac{\sinh \kappa \epsilon}{\sinh(k+1)\kappa \epsilon} \right) \right]^{1/2} \exp \left( \lim_{N \rightarrow \infty} \sum_{k=1}^{N-1} \Omega_{k-1} \right) \times \\
&\quad \exp \left\{ \frac{i}{\hbar} \left[ \frac{\sinh \kappa \epsilon}{2\epsilon} \frac{\cosh N\kappa \epsilon}{\sinh N\kappa \epsilon} (x_N^2 + x_0^2) - \frac{1}{\epsilon} \frac{\sinh \kappa \epsilon}{\sinh(k+1)\kappa \epsilon} x_N x_0 \right. \right. \\
&\quad \left. \left. - \lim_{N \rightarrow \infty} R_N x_N - \lim_{N \rightarrow \infty} S_N x_0 \right] \right\} \\
&= \left( \frac{\kappa}{2\pi i \hbar \sinh \kappa T} \right)^{1/2} \exp \left\{ \frac{i}{\hbar} \left\{ \frac{\kappa}{2 \sinh \kappa T} [\cosh \kappa T (x_b^2 + x_a^2) - 2x_b x_a] \right. \right. \\
&\quad \left. \left. - \kappa \Lambda \frac{e^{\kappa T} - 1}{e^{\kappa T} + 1} (x_b + x_a) \right\} \right\} \quad (46) \\
&\quad \left. - \kappa \Lambda \frac{e^{\kappa T} - 1}{e^{\kappa T} + 1} (x_b + x_a) \right\} \quad (47)
\end{aligned}$$

Substituting Eq. (18) into the Equation above, we have

$$\begin{aligned}
K(x_b, T; x_a, 0) &= \left( \frac{\kappa}{2\pi i \hbar \sinh \kappa T} \right)^{1/2} \exp \left\{ \frac{i}{\hbar} \left\{ \frac{\kappa}{2 \sinh \kappa T} [\cosh \kappa T (x_b^2 + x_a^2) - 2x_b x_a] \right. \right. \\
&\quad \left. \left. - \kappa \left( x_a - \frac{x_b - x_a}{e^{-\kappa T} - 1} \right) \frac{e^{\kappa T} - 1}{e^{\kappa T} + 1} (x_b + x_a) \right\} \right\} \\
&= \left( \frac{\kappa}{2\pi i \hbar \sinh \kappa T} \right)^{1/2} \exp \left\{ \frac{i}{\hbar} \left[ \frac{\kappa}{2 \tanh \kappa T} (x_b^2 + x_a^2) - \frac{\kappa x_b x_a}{\sinh \kappa T} \right. \right. \\
&\quad \left. \left. - \kappa \left( x_a - \frac{x_b - x_a}{e^{-\kappa T} - 1} \right) \tanh \frac{\kappa T}{2} (x_b + x_a) \right] \right\}. \quad (48)
\end{aligned}$$

It is evident that the propagator above tends to the free Schrödinger propagator in the limit  $\kappa \rightarrow 0$ .

Finally we perform an analysis of the time evolution in terms of the quantum propagator (48). Suppose that at the initial time we have a Gaussian wave-packet

$$\psi_0(q) \propto \exp \left( -\theta_0 q^2 + \frac{i}{\hbar} v_0 q \right),$$

, which describes a unit mass particle localized in a neighborhood of the point  $q = 0$  with an initial velocity  $v_0$ . At a later time  $t$ , the system under consideration will be characterized by the propagated wave-packet distribution, i.e.

$$\psi_T(x) = \int_{-\infty}^{\infty} K(x, T; q, 0) \psi_0(q) dq \propto \exp \left\{ -\theta_1 (x - \langle x \rangle)^2 + \frac{i}{\hbar} \theta_2 q^2 \right\}. \quad (49)$$

The direct calculation shows that the Wave-function  $\psi_T(x)$  still remains Gaussian. The mean value of position is

$$\langle x \rangle = v_0 \frac{\tanh \kappa T}{\kappa} \quad (50)$$

The mean value of velocity of the wave packet

$$\langle v \rangle = \frac{2k \tanh \left( \frac{\kappa T}{2} \right)}{e^{-\kappa T} - 1} \langle x \rangle + v_0 \quad (51)$$

The coefficient  $\theta_1$  in Eq. (49)

$$\theta_1 = \frac{\frac{\tanh^2(\kappa T)}{\hbar^2 \kappa^2}}{4 \left[ \alpha_0 - \frac{i}{\hbar} \left( \frac{\kappa}{2 \tanh \kappa T} - \kappa \tanh \frac{\kappa T}{2} - \kappa \frac{\tanh \frac{\kappa T}{2}}{e^{-\kappa T} - 1} \right) \right]} \quad (52)$$

Let us compare our result with the results obtained by Kochan's approach<sup>10,11</sup> and Caldirola-Kanai's method<sup>19</sup>, Das's method<sup>19</sup>. The results obtained by Kochan's (abbr. Koch) approach<sup>9</sup> are

$$\begin{aligned} \langle x \rangle_{Koch} &= 2v_0 \frac{\tanh \frac{\kappa T}{2}}{\kappa} \\ \langle v \rangle_{Koch} &= \frac{2v_0}{1 + e^{-\kappa T}} - \frac{3}{2} \kappa \langle x \rangle \\ \theta_{1_{Koch}} &= \frac{\frac{\kappa^2}{16\hbar^2 \tanh^2 \{0.5\kappa T\}}}{\alpha_0 - i \frac{\kappa(3 - e^{-\kappa T})}{4\hbar(1 - e^{-\kappa T})}}. \end{aligned}$$

The results obtained by Caldirola-Kanai (abbr. CK) method are

$$\begin{aligned} \langle x \rangle_{CK} &= \frac{v_0}{\kappa} (1 - e^{-\kappa T}) \\ \langle v \rangle_{CK} &= v_0 \\ \theta_{1_{CK}} &= \frac{\frac{\kappa^2}{4\hbar^2(1 - e^{-\kappa T})^2}}{\alpha_0 - i \frac{\kappa}{2\hbar(1 - e^{-\kappa T})}}. \end{aligned}$$

The results obtained by Das's (abbr. DGST) method are

$$\begin{aligned} \langle x \rangle_{DGST} &= \frac{v_0}{\kappa} (1 - e^{-\kappa T}) \\ \langle v \rangle_{DGST} &= v_0 e^{\kappa T} \theta_{1_{DGST}} = \frac{\frac{\kappa^2}{4\hbar^2(1 - e^{-\kappa T})^2}}{\alpha_0 - i \frac{\kappa}{2\hbar(1 - e^{-\kappa T})}}. \end{aligned}$$

The comparison among these results above is illustrated in Fig. 1, where  $q = 0\text{m}$ ,  $v_0 = 5\text{ms}^{-1}$ ,  $\kappa = 0.6\text{s}^{-1}$ . The curve obtained by our method is labeled with 'LG'. It is clear that the expectation value of velocity obtained by our method and Kochan's method look very

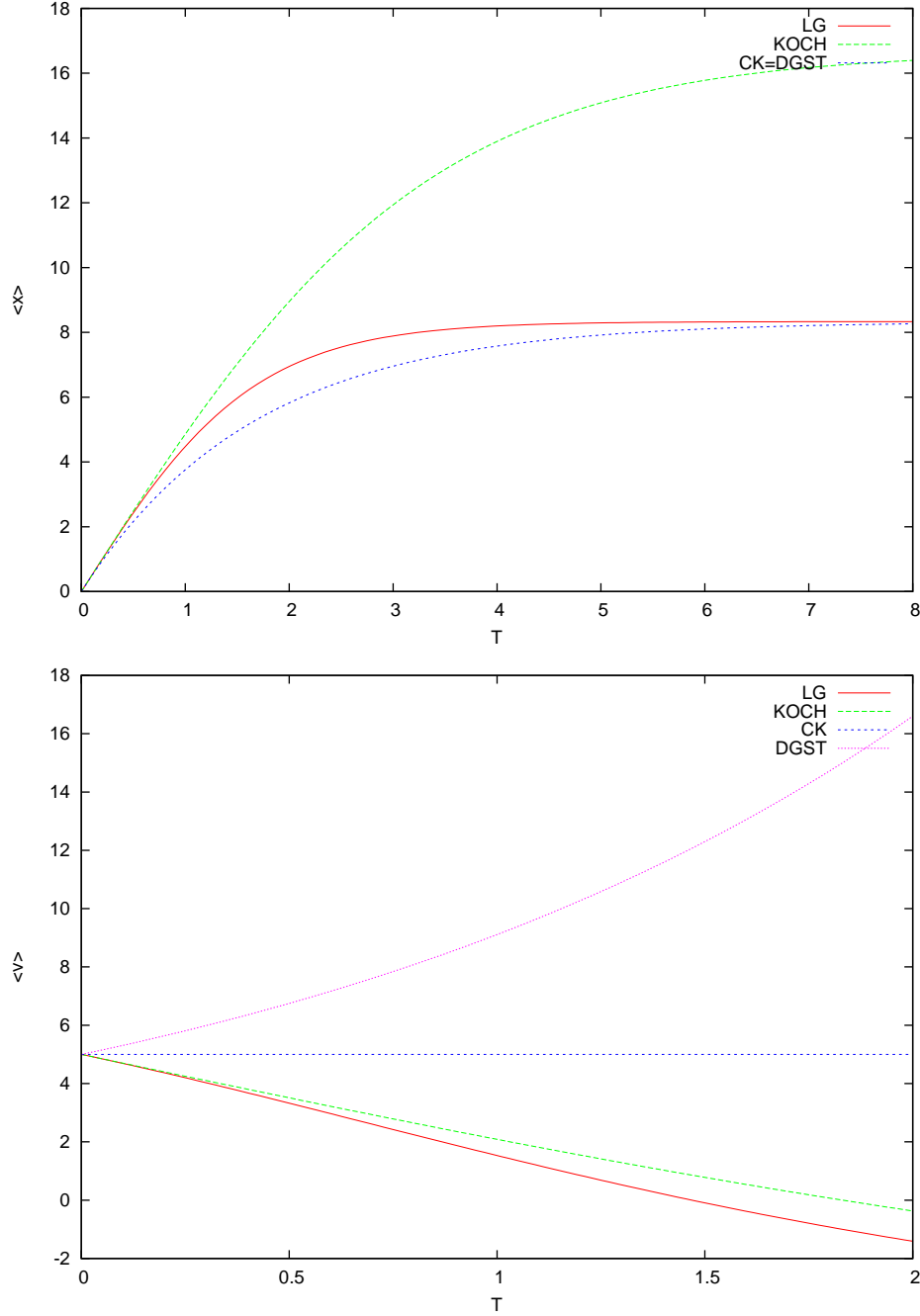


FIG. 1. Comparison of the mean values of position and velocity obtained by various methods

similar. The expectation value of position obtained by our method is quite different from that obtained by Kochan's method, but with time evolution,  $\langle x \rangle$  obtained by our method approaches  $\langle x \rangle_{CK}$ . The reason is

$$\lim_{T \rightarrow \infty} \langle x \rangle = \lim_{T \rightarrow \infty} \langle x \rangle_{CK} = \frac{v_0}{\kappa}, \quad \lim_{T \rightarrow \infty} \langle x \rangle_{Koch} = \frac{2v_0}{\kappa}.$$

Because our averaged velocity  $\langle v \rangle$  and Kochan's averaged velocity possesses a similar advantage that they decrease with respect to time translations, also possess similar restriction that our averaged velocity is reliable in the time interval  $[0, \ln(\sqrt{2} + 1)/\kappa]$  and Kochan's reliable interval is  $[0, \ln 3/\kappa]$ . Exactly at the moment  $\ln(\sqrt{2} + 1)/\kappa$ , our averaged velocity becomes zero and going over, it would produce a negative (nonphysical) mean velocity.

## V. CONCLUSION

Kochan<sup>11</sup> have compared his results with other results obtained by the method of Caldirola-Kanai and other methods. Because the aforementioned similarity, our approach possesses the advantage of Kochan's approach, but the position obtained by our approach likes that obtained by CK-method. Since both Kochan's approach and our approach are based ultimately on the classical history, this similarity exists between the approaches. For quantization of more complicated damping mechanical systems, one can utilize numerical method to obtain classical trajectory, then construct approximation of the Lagrangian density.

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